Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ELLIPTIC ORBITS AND THE GROWTH OF THE THIRD LAW WITH NEWTON.

IN my first article on Newton in this magazine[1] is quoted a memorandum written by Newton saying that in 1666 he began to think of the possibility of gravity reaching the moon's orbit. First of all he found the force along the normal to a circular orbit in a sphere due to a particle revolving within this sphere in the orbit mentioned. Having thus found what is generally known as "Huygens's theorem," $f = v^2/r$, he deduced "from Kepler's rule of the periodical times of the planets being in a sesquialterate proportion of their distances from the centers of their orbits" (that is to say, that the cubes of the latter distances are proportional to the squares of the periodic times), that the forces which keep the planets in their orbits must be inversely as the squares of their distances from the centers about which they revolve. The comparison of the distance fallen through by the moon in a short time with the distance it would fall through under a force inversely proportional to the square of its distance from the earth's surface was found to answer "pretty nearly." The same law of force would result from considering, as Newton did in 1675, the gravitating force as an emanation from the attracting body; for the areas of the surface of the spheres to be affected at greater and greater distances increase as the squares of their radii, since the area of the surface of a sphere of radius r is $4\pi r^2$, and hence the intensity of the force at any particular point on any sphere to be affected would be inversely as the square of the radius of the sphere. Further, at a later period which we have some grounds for fixing as the end of 1679 or the beginning of 1680, though it might possibly have been about two years or more before this, Newton found that under a central force varying inversely as the square of the distance from the center, a small body must revolve in an ellipse with the center of force as a focus, and with a radius drawn to that center describe areas proportional to the times. It should be noticed that the law of areas was not found in 1666, and does not seem to have been at first found for central forces in general. In this article my first purpose is to attempt a reconstruction of the Newtonian process of advancing from the not very difficult investigation of circular orbits to that of more general orbits. This seems to have occupied his thoughts in 1679 and 1680, and he had not advanced beyond this point before Halley's visit in 1684.

---

[1] *The Monist,* Vol. XXIV, 1914, p. 198.

The following is a summary of the contents of this article. Section I: The essential point in the calculation of the central force from the orbit or (by simply reversing the calculation) the orbit from the force is the finding of the distance fallen through by the attracted body toward the center of force in any infinitesimal time. From this expression the time could obviously be eliminated by Kepler's law of areas, and this law and its generalization for *any* central force were fairly obvious deductively. A somewhat simpler proof of the generalized theorem seems to have been devised, and the theorem placed in its proper logical position as first of the series on central forces. The tract which Newton wrote in 1684 and the various editions of the *Principia* (1687, 1713, 1726) have been carefully compared; and the comparison has brought to light the remarkable fact, which does not seem to have been noticed hitherto, that many changes were introduced into the edition of 1713 which give the expression of the central force in "intrinsic" coordinates. This expression had been given by Johann Bernoulli in 1710, and it is thus incorrect to attribute it to Newton (Sections II and IV). Section III is concerned with Newton's indirect method of finding the orbit from the force and with the mistaken assertion of Bernoulli and others that Newton's proof contains an error. Section V contains a contribution to the problem as to the way in which Newton arrived at his third Law.

I.

In the article referred to, we have seen that the essential thing to be determined in the case of a circular orbit was seen by Newton to be the distance fallen through in an infinitely small period of time. The same determination, as we shall shortly see, is also the essential thing in the strict calculation of orbits other than circular, and the determination in such a case is much more difficult since it requires a knowledge of technical mathematics that such men as Wren, Hooke, and Halley do not seem to have possessed.

Let us consider a small body made to revolve round a fixed center of force (S) attracting as the square of the distance from it, by being projected from a certain point with a certain velocity in a direction which is not along the line toward S. To form some idea of what the orbit is like, we will divide the whole time considered into a very large number of equal parts and work out, as any mathematician of the seventeenth century after Galileo naturally would, what would happen if the force toward S were to adjust itself by jerks at the end of each subdivision of time and were constant throughout each of these subdivisions. Suppose that the length of each of these subdivisions of time is so short that f, the numerical measure of the acceleration toward S due to the force, may be considered as constant between the positions of the body at the beginning and end of any subdivision of the time. Then, by Galileo's law, the distance fallen through toward S is $ft^2/2$, t being the length of a subdivision. Thus, as Newton showed in the sixth Proposition of the first Book of the *Principia*,[2] f is proportional to q/t^2, where q is the half versed sine which expresses the distance fallen through in the time t. Newton, at least in his later work, considered the versed sine in the middle of an evanescent arc and noticed that it points toward the center of force. Of course at that time trigonometrical functions were *lines* and not the *ratios* they became later on.

Newton's next problem was to express this infinitesimal time in terms of quantities belonging to the curve described, so that the orbit could be found from a knowledge of f, and f could be found from a knowledge of the orbit. The way to this expression would be shown by Kepler's discovery that the areas swept out by the radius vector joining a planet to the sun are proportional to the times of describing them, and thus that our t is proportional to a small triangular area whose calculation is discussed below. We will now consider whether Kepler's proposition can be deduced from general suppositions as to the motion of the small body about S.

Suppose that the body is projected at P in the direction PR under the influence of an attractive force to the fixed center S; then PR will be the tangent at the point of the orbit. If Q is a point on the orbit infinitely near P, and reached at an infinitesimal time, which we may denote by dt, after it leaves P, and QR is parallel to SP, then RQ is half the versed sine of the infinitely small arc PQV, where

---

[2] Any edition of the *Principia* will do. This reference is only made for shortness, as the method used by Newton to establish this result could not surely vary greatly from 1679 to 1713. It is on another question that the variations in Prop. VI throw light, as will be shown below.

V is a point on the orbit and QV is equal to PQ. Then RQ is equal to the diagonal of the parallelogram of adjacent sides PQ and QV, and this diagonal when produced passes through S. The diagonal mentioned is also proportional to the central force at Q. Since, now, RQ is parallel to PS, it is evident that the triangle SPQ, which is the area swept out by the radius vector in an infinitely small part of time, is equal in area to the triangle SPR. Proceeding in this way, if we imagine the original particle to proceed along the line PR unaffected by the force at S and therefore with unaltered velocity, so that equal spaces PR, RW, and so on, are described in equal times, the corresponding triangles formed by the actual orbit, SPQ, SQV, and so on, are all equal to one another and to the triangle SPR. Adding up all the infinitesimal triangles formed by the orbit, SP, and some other radius vector, and then going to the limit in the way familiar to those who worked with infinitesimals, we can conclude that the areas described by the radius vector increase proportionally to the time.

This is seen to be independent of the magnitude of the force directed to S, and indeed, from a logical point of view, it is the first proposition to be proved about central forces. Accordingly, though considerations on the centripetal force causing motion in a circle probably preceded the discovery of the theorem on the equable description of areas, we find the theorem as the first in the formally written out *Propositiones de Motu* of 1684[3] and as the first Proposition of the first Book of the *Principia* (Section II).[4] In both these places, however, the continuously acting central force is not a force acting constantly throughout each small interval of time, but is replaced by impulsive forces acting toward S at the end of each small interval. The latter method is rather simpler since we do not have to consider that the space through which the central force draws the particle in an infinitely small time is proportional to the square of the time. When the number of the infinitesimal triangles is increased indefinitely, this perimeter ultimately becomes a continuous curve and the instantaneous forces ultimately become a continuously acting force by which the body is continuously deflected from the tangents. Also in the latter case, the half versed sines mentioned above appear at once as diagonals to the parallelograms determined by consecutive arcs of the orbit. The second and third Propositions form the easily proved converse of the first.[5]

---

[3] Rouse Ball, *Essay*, pp. 36-37.

[4] The two Corollaries printed in the first edition were omitted in the second edition, and a new set of six Corollaries added, of which the first, which is that v varies inversely as p, will be mentioned subsequently.

If now we substitute in our equation for f the value of t, or dt as it was called later, in the terms of the elementary area described in the instant dt, which is expressed by the area SPQ, which is half the product of SP, or r, and the perpendicular from Q on SP, which we may call QT, we find that f is proportional to l/r^2, where l is the limiting value of QR/QT^2 when PQ is infinitely small. This quantity l is found to depend merely on the curvature of the curve, and it can be found in any particular case. The formula expressing f as varying inversely as r^2 was given as the third Proposition of the *Propositiones de Motu* of 1684[6] and also in the sixth Proposition of the first Book of the *Principia*. The second and third Corollaries to this Proposition, which first appeared in the second edition, relate to another method of finding the area of the elementary triangle SPQ which does not involve r explicitly, but the area of the triangle is expressed in terms of the perpendicular p from S on PR and the arc PQ or ds.

---

[5] There is nothing to correspond to the second and third Propositions in Newton's tract of 1684. In the second edition of the *Principia* no very important alterations were made to these propositions, but two Corollaries were added to the second.

[6] Rouse Ball, *op. cit.*, pp. 37-38.

II.

There are three things to be noticed about this second determination of the orbit. The first is that it was not given in the above-mentioned tract of 1684. The second is that in the first edition of the *Principia,* the sixth Proposition was that f is proportional to l/r^2 ("is ultimately reciprocally as the solid SP quad. × QT quad./QR"), and the Corollary simply stated that, if the orbit is given, the law of force will be found from l/r^2 in various cases in the following Propositions. In the second edition of 1713[7] the sixth Proposition was that f varies as q/t^2, and it had five Corollaries: the first gave the fact that f is proportional to l/r^2; the second, third, and fourth that f is proportional to $QR/(p.ds)^2$, and special cases; and the fifth is much as the Corollary in the first edition. The third thing is that the fact that v varies inversely as p was not explicitly noticed in the first edition of the *Principia,* but the first Corollary to the first Proposition in the second edition contains this remark. This third point is only of importance in the light of the first and second. For these reasons it seems, then, that Newton did not at first make use of the equation of the orbit which is given in terms of p and r, but only of the equation expressing that f varies directly as l and inversely as r^2. This form of the equation also seems very suggestive of the force varying inversely as r^2, and we only have to determine l and show that it does not depend on r.

In the tract of 1684, Newton determined this quantity for various curves in almost exactly the same way as he determined it in the *Principia*. In the first problem of the tract Newton solves the question of finding the law of force tending to any point in the circumference of a circle when a body revolves in the circumference.[8] This is the same as the seventh Proposition of the first Book of the first edition of the *Principia*: in the second edition this Proposition was the more general one to find the law of force to *any* given point when the body revolves in the circumference of a circle. The second problem of the tract was to find, if the body revolves in an ellipse, the law of force tending to the center of the ellipse.[9] This is the same as the tenth Proposition of the first Book of the *Principia*.[10] The third problem of the tract is: If a body revolves in an ellipse, to find the law of centripetal force tending to the focus.[11] This important Proposition is the eleventh of the first Book of the *Principia*.[12]

---

[7] It is not impossible that the second, third, and fourth Corollaries were added in consequence of Bernoulli's paper of 1710, in which the advantages of working with a (p, r) equation were first shown. Lord Brougham and E. J. Routh (*Analytical View of Sir Isaac Newton's Principia,* London, 1855, pp. 45-48, 52) unjustly speak as if Newton explicitly gave this formula before Bernoulli. On the importance of such equations cf. *ibid.*, pp. 63-64.

In this Proposition we find that the limit l referred to above is, from geometrical considerations, h^2a/b^2. When the center of force is to be at the center of the ellipse, we find that $2l$ is r^3/a^2b^2, and is therefore not independent of r. We see, then, that the centripetal force to the center of the ellipse is proportional to r.

III.

Kepler's first law is that the planets move in ellipses having the sun in one focus; so that Newton's Proposition shows that the force exerted upon them by the sun must be inversely proportional to the square of the distance. The converse of this proposition can be quite easily proved by returning backward in the argument, and this would appear to have been the way in which Newton assured himself that a body under the action of a centripetal force varying inversely as the square of the distance moves in one of the conic sections. This was stated as the first Corollary to the thirteenth Proposition of the *Principia*,[13] the twelfth and thirteenth Propositions having proved that the force tending to the focus when the body moves in a hyperbola or parabola is inversely as the square of the distance.[14]

---

[8] Rouse Ball, *op. cit.*, pp. 38-39.

[9] *Ibid.*, pp. 39-40.

[10] This proof is repeated in the second edition, but in the second edition an alternative proof, which makes use of the third Corollary of Proposition VI, is also given.

[11] Rouse Ball, *op. cit.*, pp. 40-41.

[12] An alternative proof was added in the second edition.

In both the second and third problems of the tract of 1684, an essential point in the geometrical argument is a use made of the diameter of the ellipse through the center C which is conjugate to the diameter CP. To this fact apparently refers the incident described by Conduitt after Halley left Cambridge in 1684, when Newton, trying to reproduce his calculation of the path described by a body under a force varying inversely as the square of the distance, failed to obtain his former result because "in describing an ellipse coarsely with his own hand, he had drawn the two axes of the curve instead of two conjugate diameters somewhat inclined to one another. When this mistake was corrected he obtained the result which he had announced to Halley."[15] This passage was noticed particularly by De Morgan, who said that it "carries truth on the face of it."[16]

When Newton had found that a body acted upon by a centripetal force directed to a fixed point and measured by μ/r^2 describes an ellipse about that fixed point as a focus, of such a kind that μ is proportional to a/b^2, he would find that the periodic time T is given by $2\pi ab/h$, where for h is substituted the square root of $\mu b^2/a$; πab being the known expression for the area of an ellipse. From this equation we easily find that T is proportional to $a^{3/2}$ or that T^2 is proportional to a^3. This expresses Kepler's third law that the squares of the periodic times are proportional to the cubes of the major axes; and, in the fourth theorem of the tract of 1684, Newton proved this law of Kepler and added a note on the application to planetary motion. This theorem is the same as the fifteenth Proposition of the first Book of the *Principia*,[17] "that the periodic times in ellipses are in the sesquiduplicate ratio of their major axes," but to this Proposition no addition about planetary motion was made.

---

[13] This Proposition was practically unaltered in the second edition.

[14] The only important change made in the twelfth Proposition in the second edition of the *Principia* was the addition of an alternative proof.

[15] Rouse Ball, *op. cit.*, p. 26.

[16] *Essays on the Life and Work of Newton,* Chicago and London, 1914, p. 141.

The fourth problem[18] of the tract of 1684 is: "Given that the centripetal force is inversely proportional to the square of the distance from the center of force, and knowing the quantity of that force, it is required to determine the ellipse which a body, projected from a given point with a given velocity, describes." This problem should be compared with the seventeenth Proposition of the first Book of the *Principia*:[19] "Supposing the centripetal force to be reciprocally proportional to the squares of the distance from the center, and that the absolute quantity of that force is known; it is required to determine the line which a body will describe that is let go from a given place with a given velocity in the direction of a given right line." In 1684 the motion of the body P was compared with the *circular* orbit of another body revolving under the same force: in 1687 the body P was compared with a body *p* revolving in *any given* orbit. In 1684 P was assumed to describe an ellipse: in 1687 it was assumed to describe any conic section. Indeed, in the tract of 1684, the possibility of P describing a parabola or a hyperbola was only referred to at the end of this fourth problem, whereas parabolic and hyperbolic motion were proved to arise from a force varying inversely as the square of the distance in the thirteenth and twelfth Propositions respectively of the *Principia,* and, as has been mentioned, the first Corollary of the thirteenth Proposition indicates that conversely P moves in one of the conic sections about a focus from which proceeds a force of the kind mentioned. Lastly, in the tract of 1684, there was added to this fourth problem a note on cometary paths "which," according to Rouse Ball,[20] "is obscure and partly incorrect."

---

[17] Unaltered in all editions.

[18] Rouse Ball, *op. cit.*, pp. 43-44.

[19] In the second edition, some words of explanation were added at the end of the demonstration, possibly in view of the criticism by Johann Bernoulli in 1710. This criticism certainly directed attention to an obscurity of Newton: the obscurity did not happen to hide an error, in view of Prop. XIII, Cor. I (to which also a fuller explanation was added in the second edition). Still, Brougham and Routh (*op. cit.*, pp. 58, 62) treat Bernoulli's contention with rather unnecessary harshness. It may be added that, except for the addition just mentioned and a scholium on motion in a conic under a force to any point, that was added to the second edition, Prop. XVII remained unchanged in all editions of the *Principia*.

The rest of the tract of 1684 is devoted to a determination of the space through which a body will fall in a given time toward a center of attraction varying inversely as the square of the distance, and some determinations of the motions of projectiles in resisting mediums.

We may mention here that, among Newton's manuscripts in the Portsmouth Collection at Cambridge, there is[21] a fragment in which fluxions are employed in finding the centripetal force in an orbit. This seems to afford additional evidence that the method of fluxions was used in the discovery of the theorems afterward proved in a more Euclidean manner in the *Principia.*

[20] *Op. cit.,* p. 34.

[21] *Portsmouth Catalogue,* p. 4.

IV.

In my first article on Newton in this magazine, there is[22] an investigation of the law of force for an orbit which is not circular. It does not now seem to me very probable that the first great mathematical advance of Newton lay in satisfying himself by infinitesimal considerations that he could pass from the simple case of circular orbits to the general case. It must always be remembered that we have very little documentary evidence as to Newton's process of discovery, and that thus a reconstruction of this process can very often claim only a greater or less degree of probability. While, then, it does not seem to me likely that Newton would have been content with assuring himself in the above way that the law of the inverse square held for non-circular orbits, it seems to have been the case—as was indeed natural—that the infinitesimal elements of an orbit were considered on occasion as circular. Thus Newton worked with the versed sine of any arc.

In an infinitesimal interval, the body in all the cases which, if we are not bent on devising cases in which the curvature of the orbit becomes infinite or irregular, usually occur to us describes a circular arc about the center of curvature at the point of the curve which we consider, and Newton may possibly thus have assured himself, at least until he had strictly calculated the shape of orbits described under a given force, that the same law of force which holds for circular orbits also holds for the other orbits that occur in nature. Such an investigation may have taken place even as early as 1666, since we know that Newton was occupied in the early days of his discovery of the method of fluxions with determining the "radius of curv·ity" at any point of a curve.

But if Newton had found the expression v^2/ϱ for this normal force, then, since we have at once the additional expression fp/r for this force, he would have arrived at the equation

---

[22] *The Monist,* Vol. XXIV, 1914, pp. 211-213.

$$f = rv^2/p\varrho.$$

Possibly also Newton found that f is proportional to q/dt^2 and that q varies as ds^2 from this last equation, as stated in Lemma XI of Section I of the first Book.

Now, by the law of the equable description of areas, $p.ds$ is proportional to dt, or, as it is usually written nowadays, $pv = h$. Substituting for v in the above equation f, we get

$$f = h^2r/p^3\varrho,$$

which is practically what Newton stated in the second and third Corollaries of his sixth Proposition in the second edition of the *Principia* (1713). This formula was first given by Johann Bernoulli in 1710; and since, as Newton announced, ds^2/q is the measure of that chord of the circle of curvature at P which is drawn from P along the line PS, we see that f is inversely proportional to this chord multiplied by p^2. We can find this chord (c) by observing that $c/2p = dr/dp$; but Newton did not indicate this formula.[23] This formula allows us to conclude that the radius of curvature is $r.dr/dp$,[24] and hence to give the simple analytical expression to the third Corollary of Newton, or the expression of Bernoulli:

$$f = h^2dp/p^3dr.$$

The first Corollary to the first Proposition of the first Book was, as I have mentioned above, added only in the second edition (1713) of the *Principia*; it is that the velocity of a body attracted toward a fixed center varies inversely as p. That this simple deduction from the law of areas was not mentioned in the first edition is perhaps an additional indication that Newton did not at first work with p and r as coordinates. If he had done so, he would, from the expression v^2/ϱ for the normal force, at once have found that f varies directly as r and inversely as $p^3\varrho$, as Bernoulli found in 1710.

---

[23] Brougham and Routh (*op. cit.*, pp. 45, 46) seemed, however to state this.

[24] This expression seems to have been given first by Maclaurin in his *Treatise of Fluxions*: Maclaurin of course used the fluxional notation.

In the ellipse, we can find by the analytic geometry of conic sections that ϱ is k^2/p or k^3/ab, where k is half the diameter through the center C that is conjugate to CP, and a and b are the semi-major axis and semi-minor axis. Since we also have $p=br/k$, we have, by substitution in the above expression for f, the result

$$f=h^2a/b^2r^2.$$

This section has been devoted to a formula for f for which the credit of priority seems undoubtedly to belong to Johann Bernoulli. In the same memoir Bernoulli gave a direct proof that a force inversely as the square of the distance brings about a motion satisfying Kepler's laws. However, Newton had already proved this[25] though it must be admitted that Newton's proof was neither direct nor wholly free from obscurities.

Such equations as $f=v^2/\varrho$, which seem to have been first given explicitly by Euler in 1736,[26] are in what are called "intrinsic" or "natural" coordinates. I will discuss them on another occasion.

V.

We have seen that in the Royal Society tract of 1684 the motion of a body on a fixed center of force was considered. But this case of an immovable center does not occur in nature, for we cannot be certain that the center of attraction in nature is immovable. In the case of the sun, the center appears, according to Copernicus, to be stationary; but, in the case of the moon revolving round the earth, the center of attraction is not stationary but revolves in an elliptical orbit round the sun. It would seem a matter of instinctive knowledge that when we have to do with a body revolving round a very much larger one, the larger body is almost at rest. It is not a very far step to the supposition that only the center of gravity of bodies is at rest. But in the case of the solar system, we cannot say where this center of gravity is. However, it seems quite beyond doubt that Newton, from his early days, took the very natural view that what was decisive in the measure of gravity was the size and density of, or "quantity of matter in," a body. This quantity of matter would of course be a property of the body itself, and would remain the same whether or not any force of gravitation acted on it.

---

[25] This fact seems to have been overlooked by Paul Stäckel in his article of 1908 on "Elementare Dynamik der Punktsysteme und starren Körper" (*Encykl. der math. Wiss.*, Vol. IV, Part 1, p. 494).

[26] *Ibid.*, p. 463.

In a manuscript draft of 1684 which is in the Library of the University of Cambridge, England, which has been described by Rouse Ball,[27] and which I have examined, there is an additional law, expressing the consequence of the third law of motion of the *Principia,* which is of particular value in dealing with the motions of two or more mutually attracting bodies. It is that the common center of gravity of bodies does not change its state of motion or rest by the mutual actions of these bodies. This law, in the case of two bodies, is identical with the third law of motion, but in the case of three or more bodies the third law cannot be deduced from it, though it can be from the third law. In fact, if m and m' are two masses which are supposed to be alone in space, then the principle of the conservation of the motion of the center of gravity gives three equations (one for each rectangular axis) of the form

$$m(d^2x/dt^2) + m'(d^2x'/dt^2) = 0,$$

---

[27] *Op. cit.,* pp. 51-56.

and thus says that the accelerations are in the inverse ratio of the masses. But if there are three mass-points, we get three equations of the type of the above one, but the left-hand member of each is formed by adding three terms instead of two. Logically, we cannot deduce the mass-ratios m/m' and m'/m'' from these equations, for $a + b = 0$ does not imply that a and b are each zero; although in the present case it would seem to be a very natural thing to say that m/m' would remain the same if m'' were annihilated.

The motions of two mutually attracting bodies were dealt with in the fifty-seventh to the sixty-third Propositions of the first Book (Section XI) of the *Principia.*[28] The first of these Propositions is that two attracting bodies describe similar figures about their center of gravity and about each other. The next Proposition is that if two attracting bodies revolve about their center of gravity, then under the same forces a similar and equal orbit might be described about either one of the bodies if it were fixed. The Corollaries to this Proposition show that the results of the first, tenth, eleventh, twelfth, and thirteenth Propositions of the first Book are applicable to such motions. The sixty-first Proposition is that two bodies acted upon by their mutual attractions and not otherwise agitated or obstructed, will move as if neither attracts the other but both are attracted according to the same law of force by a third body placed in their common center of gravity.

PHILIP E. B. JOURDAIN.

---

[28] These Propositions are practically unchanged in all three editions of the *Principia.*